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An Acceleration Method for the Power Series of Entire Functions of Order 1

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Abstract. When $f(z)$ is given by a known power series expansion, it is possible to construct the power series expansion for $f(z;\, p) = e^{-pz}f(z)$. We define p_{opt} to be the value of p for which the expansion for $f(z;\, p)$ converges most rapidly. When $f(z)$ is an entire function of order 1, we show that p_{opt} is uniquely defined and may be characterized in terms of the set of singularities $z_i = 1/\sigma_i$ of an associated function $h(z)$. Specifically, it is the center of the smallest circle in the complex plane which contains all points σ_i.

1. Introduction. In this paper we present a method for accelerating the convergence of a power series expansion. The method is designed for entire functions of order 1 and finite type. That is, the coefficients α_j in the expansion

$$f(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots \tag{1.1}$$

decay sufficiently rapidly that the associated expansion

$$h(f;\, z) = \alpha_0 + \alpha_1 1!z + \alpha_2 2!z^2 + \cdots \tag{1.2}$$

converges for small z and has a finite radius of convergence R. The method consists of choosing a parameter p and reexpressing $f(z)$ in the form

$$f(z) = e^{pz}f(z;\, p), \tag{1.3}$$

where

$$f(z;\, p) = \alpha_0(p) + \alpha_1(p)z + \alpha_2(p)z^2 + \cdots. \tag{1.4}$$

On comparing the coefficient of z^k in the identity $f(z;\, p) = e^{-pz}f(z)$, it follows immediately that the coefficients are given by

$$\alpha_k(p) = \sum_{j=0}^{k} \frac{(-p)^j}{j!}\alpha_{k-j}. \tag{1.5}$$

The question we deal with in this paper is that of choosing a value p_{opt} of p which optimizes the ultimate rate of convergence of the series (1.4).

In the remainder of this section, we state the criterion we shall use for deciding whether one series converges faster than another.

In Section 2 we describe the theory as it applies to an entire function of order 1. We obtain a characterization of p_{opt} as the center of the smallest circle containing a set of points σ_i in the complex plane, where $1/\sigma_i$ is the set of singularities of $h(f;\, z)$.

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We illustrate the theory with several examples. Some of the properties of p_{opt} are merely stated in Section 2 and proved later in Section 3. In Section 4 we briefly discuss the use of this technique in cases where $f(z)$ is not an entire function of order 1 and show that it is unlikely to be effective. Finally in Section 5 we make some brief comments about determining p_{opt} numerically.

In this paper we employ the following criterion to compare rates of convergence of power series. We introduce a comparison sequence ϕ_j, $j = 1, 2, \ldots$, having the following property

$$\text{(1.6)} \qquad \phi_j > 0 \quad \forall j > J; \quad \text{either } \phi_{j+1} \geq \phi_j \quad \forall j > J, \quad \text{or} \quad \phi_{j+1} \leq \phi_j \quad \forall j > J,$$

where J is finite. Thus ϕ_j is a sequence of positive terms which is ultimately either monotonic nonincreasing or monotonic nondecreasing. Then we may compare the rates of convergence of

$$\text{(1.7)} \qquad f_1(z) = \sum \alpha_n z^n \quad \text{and} \quad f_2(z) = \sum \beta_n z^n$$

by comparing

$$H_1(z) = \sum \phi_n^n \alpha_n z^n \quad \text{and} \quad H_2(z) = \sum \phi_n^n \beta_n z^n,$$

where ϕ_j are the elements of any comparison sequence. It is generally possible to choose the comparison sequence so that either one or both of $H_1(z)$ and $H_2(z)$ have finite nonzero radii of convergence

$$1/R_1 = \limsup_{n \to \infty} \phi_n |\alpha_n|^{1/n}; \qquad 1/R_2 = \limsup_{n \to \infty} \phi_n |\beta_n|^{1/n}.$$

It is well known that when two power series have different radii of convergence, the one having the higher radius of convergence converges more rapidly. Consequently we adopt the following partial definition:

Criterion 1.8. If there exists a comparison sequence ϕ_j (satisfying (1.6) above) for which

$$\limsup_{n \to \infty} \phi_n |\alpha_n|^{1/n} > \limsup_{n \to \infty} \phi_n |\beta_n|^{1/n},$$

then the power series $\sum \beta_n z^n$ converges more rapidly than the power series $\sum \alpha_n z^n$. This somewhat crude criterion is sufficient for the theory covered in this paper.

2. Theoretical Development for Entire Functions of Order 1. In this section we shall assume that $f(z)$ is an entire function of order 1, given by (1.1), and describe a theoretical method for determining p_{opt}. Corresponding to $f(z)$ and $f(z; p)$, we define associated functions

$$\text{(2.1)} \qquad h_\nu(f; z) = \sum_{j=0}^{\infty} \alpha_j (j + \nu)! z^j, \qquad |z| < R(f; 0),$$

$$\text{(2.2)} \qquad h_\nu(f; z; p) = \sum_{j=0}^{\infty} \alpha_j(p)(j + \nu)! z^j, \qquad |z| < R(f; p).$$

Here ν is any real number for which $\alpha_j(j + \nu)!$ is finite for all j. It is shown in Section 4 that, under the assumption on $f(z)$ mentioned above, the series in (2.1) and (2.2) converge for sufficiently small z and have finite radii of convergence as indicated. Moreover, these radii of convergence are independent of ν.

The function $h_\nu(f; z; p)$ is defined by series (2.2) for values of z satisfying $|z| < R(p)$. It is defined for other values of z by analytic continuation. When there are branch cuts they are to be located between the branch singularities at $z = s_1$ and $z = s_2$ in such a way that, when z is a point on the branch cut, $1/z$ is on a straight line connecting $1/s_1$ to $1/s_2$.

If we are given $p_1 \neq p_2$ and $R(f; p_1) > R(f; p_2)$, it follows from Criterion 1.8 with $\phi_j = ((j+\nu)!)^{1/j}$ that the series (1.4) for $f(z; p_1)$ converges faster than the series for $f(z; p_2)$. Consequently, p_{opt}, the value of p for which the series for $f(z; p)$ converges most rapidly, may be characterized as the value of p which maximizes the radius of convergence $R(f; p)$ of the series (2.2) for $h_\nu(f; z; p)$.

Definition 2.3. When $f(z)$ is an entire function of order 1, p_{opt} and $R_{\text{opt}}(f)$ are the unique quantities which satisfy

$$R_{\text{opt}}(f) = R(f; p_{\text{opt}}) > R(f; p) \qquad \forall p \neq p_{\text{opt}}. \tag{2.3}$$

It is shown in Section 3 that these quantities are unique.

The problem of determining the radius of convergence of $h_\nu(f; z; p)$ may at first sight seem difficult. However, in some trivial and nontrivial cases it may be accomplished if $h_\nu(f; z)$ is a function, the location of whose singularities is known. To show this we first establish the following

THEOREM 2.4.

$$h_\nu(f; z; p) = \frac{1}{(1+pz)^{1+\nu}} h_\nu\left(f; \frac{z}{1+pz}\right). \tag{2.4}$$

Proof. This is established by series manipulation. When $|pz| < 1$, we may use (2.1), together with the expansion

$$(1+pz)^{-\lambda} = \sum_{l=0}^{\infty} (-pz)^l \binom{\lambda+l-1}{l} \tag{2.5}$$

to express the right-hand side of (2.4) successively in the forms

$$\begin{aligned} \sum_{j=0}^{\infty} \alpha_j (j+\nu)! z^j (1+pz)^{-j-\nu-1} &= \sum_{j=0}^{\infty} \alpha_j (j+\nu)! z^j \sum_{l=0}^{\infty} (-pz)^l \binom{j+\nu-l}{l} \\ &= \sum_{k=0}^{\infty} z^k (k+\nu)! \sum_{l=0}^{k} (-p)^l \alpha_{k-l}/l!. \end{aligned} \tag{2.6}$$

In view of (1.5) and (2.2), this is identical with the left-hand side of (2.4). The result follows for all p, z by analytic continuation. □

This result allows us to specify the locations of the singularities of $h_\nu(f; z; p)$ in terms of locations $1/\sigma_i$ of the singularities of $h_\nu(f; z)$. To this end we define a set $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$.

Definition 2.7. $\sigma_i \neq 0$ is an element of Σ if and only if $1/\sigma_i$ is a singularity of $h_\nu(f; z)$ in the finite part of the complex plane.

$\sigma_i = 0$ is an element of Σ if and only if $z^{\nu+1} h_\nu(f; z)$ has a singularity at infinity.

Note that since the radius of convergence of $h_\nu(f; z)$ is $R(f; 0)$,

$$\sigma_i \leqslant 1/R(f; 0) \quad \forall i. \tag{2.7}$$

When $h_\nu(f; z)$ has branch cuts, σ_i may depend on i continuously.

THEOREM 2.8. *For $p \neq 0$, the singular points of $h_\nu(f; z; p)$ in the finite part of the complex plane are $z_i = -1/(p - \sigma_i)$ for all elements $\sigma_i \in \Sigma$.*

Proof. In view of (2.4), corresponding to each singularity of $h_\nu(f; z)$ where $|\sigma_i| > 0$, there is a singularity of $h_\nu(f; z; p)$ at $z = z_i$ where $z_i/(1 + pz_i) = 1/\sigma_i$. This gives $z_i = -1/(p - \sigma_i)$. Moreover $h_\nu(f; z; p)$ is regular at all other values of z in the finite part of the complex plane except possibly at $z = -1/p$.

The situation at this point is clarified by rewriting (2.4) in the form

$$h_\nu(f; z; p) = \frac{1}{z^{\nu+1}}\left(\frac{z}{1+pz}\right)^{1+\nu} h_\nu\left(f; \frac{z}{1+pz}\right). \tag{2.8}$$

Clearly $h_\nu(f; z; p)$ has a singularity at $z = -1/p$ if and only if $z^{1+\nu}h_\nu(f; z)$ has singularity at infinity. Note that we used here the circumstance that $p \neq 0$. □

The radius of convergence $R(f; p)$ of the series for $h_\nu(f; z; p)$ coincides with the distance of the nearest singularity of $h_\nu(f; z; p)$ to the origin. Thus

$$R(f; p) = \min_{\Sigma}\left|\frac{1}{p - \sigma_i}\right|. \tag{2.9}$$

Finally, p_{opt} is defined according to (2.3) as the choice of p which maximizes $R_\nu(f; p)$. Thus

$$R_{\text{opt}}(f) = R(f; p_{\text{opt}}) = \max_p R(f; p) = \max_p \min_{\Sigma}\left|\frac{1}{p - \sigma_i}\right|. \tag{2.10}$$

This has an elegant geometric interpretation: We recall that $h_\nu(f; z)$ has a finite radius of convergence $R(f; 0)$. Thus none of the singularities $1/\sigma_i$ of $h_\nu(f; z)$ has modulus less than $R(f; 0)$, so

$$|\sigma_i| \leq 1/R(f; 0). \tag{2.11}$$

The specification (2.11) may be written in dual form as

$$\begin{aligned}(R_{\text{opt}}(f))^{-1} = (R(f; p_{\text{opt}}))^{-1} &= \min_p (R(f; p))^{-1}, \\ &= \min_p \max_{\Sigma} |p - \sigma_i| .\end{aligned} \tag{2.12}$$

Clearly then p_{opt} is the value of p which minimizes the function $\max_i |p - \sigma_i|$. In other words we have:

LEMMA 2.14. *p_{opt} is the center of the smallest circle in the complex plane whose closure includes all points σ_i contained in Σ.*

In Section 3, we shall determine several general properties of p_{opt} based on this geometric characterization. For example, we shall show that p_{opt} is uniquely defined; that when the coefficients α_j of the expansion of $f(z)$ are all real then p_{opt} is also real. In addition, when $f(z)$ is of definite parity, that is either $f(z) = -f(-z)$ or $f(z) = +f(-z)$ for all z, then $p_{\text{opt}} = 0$.

We conclude this section by presenting some examples in which the set Σ is small enough to identify p_{opt} by inspection. Thus, when Σ contains only one element σ_1, Lemma 2.14 gives $p_{\text{opt}} = \sigma_1$, and when Σ contains only two elements σ_1 and σ_2, p_{opt}, being the center of the smallest circle through these two points, is clearly equal to $\frac{1}{2}(\sigma_1 + \sigma_2)$.

Some examples are listed in Table 2.13. There the series is indicated by its coefficients in column 2, and the functional form if known in column 1. The calculation of p_{opt} depends on the choice of ν, but the result, in column 6, is independent of ν. In simple examples like these, it is often possible to find a simple functional form for $h_\nu(f; z)$ for some value of ν even though no simple functional form for $f(z)$ is available.

TABLE 2.13

	$f(z)$	α_k	ν	$h_\nu(f;z)$	Σ	ρ_{opt}	$f(z,\rho_{\text{opt}})$
(1)	$z^\ell e^z$	$\frac{1}{(k-\ell)!}$	0	$\frac{\ell! z^\ell}{(1-z)^{\ell+1}}$	1	1	z^ℓ
			$-\ell$	$z^\ell/(1-z)$	1		
(2)	$\frac{e^z-1}{z}$	$\frac{1}{(k+1)!}$	0	$-\frac{1}{z}\ln(1-z)$	all $\sigma \in [0,1]$	$\frac{1}{2}$	$\frac{\sinh \frac{1}{2} z}{z}$
			1	$\frac{1}{(1-z)}$	0,1		
(3)	$\frac{1}{k(k!)}$	$k \geq 1$	0	$-\ln(1-z)$	0,1	$\frac{1}{2}$	
(4)		$\frac{1}{(k+\frac{1}{2})!}$	$\frac{1}{2}$	$\frac{1}{(1-z)}$	0,1	$\frac{1}{2}$	
(5)	$(-1)^{\frac{k+1}{2}}/(k(k!))$ 0	k odd k even	0	$\frac{1}{2i}\ln\left(\frac{1-iz}{1+iz}\right)$	$-i, i, 0$	0	
(6)	$-\frac{1}{(\sqrt{2})^{k+1}k!}\sin(k-1)\,\pi/4$		0	$\frac{1-z}{(z-1)^2+1}$	$1+i,\ 1-i$	$\frac{1}{2}$	

This list can be extended by simple scaling. For example when

$$F(z) = \kappa f(\lambda z), \qquad \kappa, \lambda > 0,$$

then $h_\nu(F; z) = \kappa h_\nu(f; \lambda z)$ and

$$p_{\text{opt}}(F) = p_{\text{opt}}(f)/\lambda. \tag{2.14}$$

Care should be taken in other linear processes. For example, when

$$F(z) = \sum f_i(z) \quad \text{and} \quad h_\nu(F; z) = \sum h_\nu(f_i; z),$$

$p_{\text{opt}}(F)$ is not linearly related to $p_{\text{opt}}(f_i)$. It is however simple to show that

$$p_{\text{opt}}(F) \geqslant \min_i p_{\text{opt}}(f_i). \tag{2.15}$$

Some of these examples illustrate general results. Thus Example (1) verifies that the method works as expected for a polynomial. In Example (2) we see that

$f(z;\, p_{\text{opt}})$ is an even function of z. There is a simple theorem to the effect that if the set $f(z;\, p)$ includes an even function or an odd function, then this is the member of the set which has the most rapid rate of convergence. This theorem follows immediately from another, illustrated in (5), to the effect that when $f(x)$ is even or odd, $p_{\text{opt}} = 0$. This is proved in Section 3.

Example (6) is one which arose in the practical calculation (Gabutti [2]) of integrals of the form

$$I(\phi;\, \omega) = \int_0^\infty e^{-t^2} J_0(\omega t)\phi(t^2)t\, dt \tag{2.16}$$

for various values of ω and various functions ϕ. Using known properties of the Bessel function, one may expand $\phi(t^2)$ in a Laguerre series expansion. Setting $z = \omega^2/4$, we find

$$I\left(\phi;\, 2\sqrt{z}\right) = \frac{1}{2} e^{-z} \sum_{k=0}^{\infty} \alpha_k z^k, \tag{2.17}$$

where α_k is the Laguerre expansion coefficient

$$\alpha_k = \frac{1}{k!} \int_0^\infty e^{-t} L_k(t)\phi(t)\, dt. \tag{2.18}$$

The entry in Example (6) is the value of this coefficient when $\phi(t) = \sin t$.

3. Further Results. In Section 2, we showed that p_{opt} could be determined as the minimizer of a function $R(f;\, p)^{-1}$

$$R(f;\, p)^{-1} = \max_{\Sigma} |p - \sigma_i|, \tag{3.1}$$

where Σ includes all values σ_i for which the function $h_\nu(f;\, z)$ has singularities at $z = 1/\sigma_i$, together with $\sigma_i = 0$ when $z^{\nu+1}h_\nu(f;\, z)$ has a singularity at infinity. This section is devoted entirely to a discussion of the nature of $R(f;\, p)^{-1}$ in terms of a given set Σ. The principal results of this section are Theorems 3.8, 3.11, and 3.12 below.

First we recall that $h_\nu(f;\, z)$ has a finite nonzero radius of convergence $R(f; 0)$. Thus all elements σ_i satisfy

$$|\sigma_i| \leqslant 1/R(f; 0) \tag{3.2}$$

and so lie in a finite region of the complex plane. Next we recall that $h_\nu(f;\, z)$ may have branch singularities. However, we stated in Section 2 that any branch cut should be located so that, when $z = 1/\sigma'$ lies on the branch cut, σ' should lie on a straight line connecting the corresponding branch singularities $z = 1/\sigma_1$, $z = 1/\sigma_2$. With the branch cut arranged in this way, it is clear that, for all σ', $|p - \sigma'| \leqslant \max |p - \sigma_1|, |p - \sigma_2|$. So when evaluating $R(f;\, p)^{-1}$ given by (3.1) above, only branch singularities and poles need be included. Thus

$$R(f;\, p)^{-1} = \max_{\Sigma'} |p - \sigma_i|, \tag{3.3}$$

where Σ' is a finite subset of Σ which omits singularities on branch cuts other than the ones at the terminations of the branch cuts, i.e., the branch singularities.

Theorems 3.4 and 3.9 below are geometric in nature. Let the elements of Σ' be represented in the complex plane by $S_1, S_2, \ldots$ and p by P. Then $R(f;\, p)^{-1}$ is the radius of the smallest circle having center at P which passes through or contains all the points S_i.

THEOREM 3.4. *Let l be any line in the complex plane and P a point on this line, parametrized by its distance t from an origin* 0 *on this line. Let*

$$F(t) = \max_i PS_i.$$

Then $F(t)$ is a convex downward continuous function of z whose right-hand derivative is a piecewise continuous monotonic increasing function of t.

Proof. Let D_i be the foot of the perpendicular from S_i onto l, and let $S_iD_i = d_i$ and $0D_i = l_i$. Clearly

$$(PS_i)^2 = d_i^2 + (l_i - t)^2 \tag{3.4}$$

and $F(t) = \max_i \phi_i(t)$, where

$$\phi_i(t) = \sqrt{d_i^2 + (l_i - t)^2}\,. \tag{3.5}$$

First we note that $\phi_i(t)$ is convex downward; that is, $\phi_i'(t)$ is monotonic increasing in t. Moreover, when S_i and S_j refer to two distinct points, $\phi_i(t)$ and $\phi_j(t)$ are distinct curves which intersect either once (when $l_i \neq l_j$) or not at all (when $l_i = l_j$). If they do intersect say at t_{ij}, then $\phi_i'(t_{ij}) \neq \phi_j'(t_{ij})$. Thus the curve

$$F(t) = \max_i \phi_i(t) \tag{3.6}$$

is continuous and its derivative is piecewise continuous. At points between these discontinuities $F(t)$ coincides with some $\phi_i(t)$ and so, between the discontinuities $F'(t)$, the right-hand derivative is monotonic increasing.

Let $\bar{t}$ be a point on $F(t)$ where there is a discontinuity. Since $F(x) = \max_i \phi_i(t)$ clearly in some ε neighborhood of $\bar{t}$,

$$F(t) = \phi_R(t) > \phi_L(t), \quad t \in (\bar{t}, \bar{t} + \varepsilon),$$
$$F(t) = \phi_L(t) > \phi_R(t), \quad t \in (\bar{t} - \varepsilon, \bar{t}).$$

Since $\phi_L(t)$ intersects $\phi_R(t)$ at $t = \bar{t}$, it follows that $\phi_R'(\bar{t}) > \phi_L'(\bar{t})$, and since the derivatives of both $\phi_R(t)$ and $\phi_L(t)$ are monotonic increasing, we have

$$F'(t - \varepsilon) = \phi_L'(t - \varepsilon) < \phi_L'(\bar{t}) < \phi_R'(\bar{t}) = F'(\bar{t}) < \phi_R'(\bar{t} + \varepsilon) = F'(t + \varepsilon).$$

This establishes the conclusion of Theorem 3.4, namely that $F(t)$ is a convex downward function as its right-hand derivative is monotonic increasing for all t. □

COROLLARY 3.7. *The function $F(t)$ of Theorem* 3.4 *has a unique minimum.*

COROLLARY 3.8. *The function $R^{-1}(f;\, p)$ has a unique minimum and no maxima nor saddle points.*

If this were not true, one could construct a line (either through two distinct minimum points or through a saddle point) which violated the result of Theorem 3.4.

This minimum value of p is denoted by p_{opt}.

THEOREM 3.9. *If the points S_i are located in such a way that there is an axis of symmetry, then p_{opt} lies on this axis of symmetry.*

Proof. Let l_+ be the axis of symmetry and P be a point not on this axis. Let l be a line perpendicular to l_+ which intersects l_+ at 0. Then the function $F(t)$ of Theorem 3.4 for l is clearly symmetric about 0. However, a symmetric convex downward function has its minimum at the point of symmetry. Thus, on the line l, $F(t)$ has its minimum at 0 and so, for $t \neq 0$, $F(t) > F(0)$. This establishes that p_{opt} cannot be off the axis of symmetry. □

THEOREM 3.10. *If the points S_i are located in such a way that there is a point of symmetry* 0, *then p_{opt} is the point of symmetry.*

Proof. Let p be any point other than the point of symmetry 0, and let l be the line containing P and 0. The function $F(t)$ of Theorem 3.4 for l is clearly symmetric about 0. Since $F(t)$ is convex downward, and symmetric, it has its minimum at 0. Thus $F(t) > F(0)$ giving $R(f;\, p)^{-1} > R(f;\, p(0))^{-1}$, and it follows that $p(0)$, the value of p at the point of symmetry, is p_{opt}. □

We now relate these results to the original function $f(z)$.

THEOREM 3.11. *When the coefficients α_j are all real, then p_{opt} is real.*

Proof. When the coefficients α_j are all real, the singular points of $h_\nu(f;\, z)$ are either real or occur in complex conjugate points. Thus the point configuration S_1, $S_2, \ldots$ is symmetric about the real axis. In this case, it follows from Theorem 3.9 that p_{opt} lies on the real axis and so is real.

THEOREM 3.12. *When the function $f(z)$ is symmetric or antisymmetric, i.e., $f(z) = \pm f(z)$ for all z, then p_{opt} is zero.*

Proof. In this case the function $h_\nu(f;\, z)$ is symmetric or antisymmetric and so its singularities are symmetrically located with respect to the origin. This theorem is then a direct consequence of Theorem 3.10.

The following is a direct corollary.

COROLLARY 3.13. *When $f(z) = e^{qz}g(z)$ and $g(z)$ is symmetric or antisymmetric then $p_{\text{opt}} = q$.*

4. Remarks about Functions not Entire of Order 1. In this section, we discuss the reasons why the series acceleration technique is not likely to be effective unless $f(z)$ is an entire function of order 1. We shall also show that the radius of convergence $R(f;\, p)$ of $h_\nu(f;\, z;\, p)$ defined in (2.2) is independent of ν. The discussion is based on the order-type classification of entire functions discussed in Hille [1, pp. 182–233]. In terms of

$$M(r;\, f) = \max_{|z|=r} |f(z)|\,, \tag{4.1}$$

the entire function $f(z)$ is of order μ if

$$\limsup_{r\to\infty} \frac{\log\log M}{\log r} = \mu, \tag{4.2}$$

and if $0 < \mu < \infty$, it is of type τ if

$$\limsup_{r\to\infty} r^{-\mu} \log M = \tau. \tag{4.3}$$

When μ is a positive integer and $p(z)$ is a polynomial, the function $f(z) = p(z)\exp(\tau z^{\mu})$ is of order μ and type τ. We shall not discuss cases where μ is infinite, or τ is zero or infinite. The results we shall require are the following.

When both μ and τ are finite and nonzero, the rate of increase of the derivatives $\alpha_j j!$ of $f(z)$ is given by

$$\limsup_{n\to\infty} n\,|\,\alpha_n\,|^{\mu/n} = e\mu\tau. \tag{4.4}$$

When $f(z)$ is not entire

$$\limsup_{n\to\infty} |\,\alpha_n\,|^{1/n} = 1/R_f, \tag{4.5}$$

where R_f is the radius of convergence of the power series expansion of $f(z)$.

When $f_1(z)$ and $f_2(z)$ are entire functions of order μ_1 and μ_2, respectively, and $\mu_2 > \mu_1$, then the product $f_1(z)f_2(z)$ is an entire function of order μ_2.

A very simple argument, based on (4.4), (4.5), and Criterion 1.8, establishes a familiar hierarchy for the ordering of series in terms of rapidity of convergence of the power series. Briefly, an entire function converges more rapidly than one which is not entire. Of two entire functions, the one having lowest order μ converges faster, and, when they have the same finite nonzero order, the one having lowest type τ converges faster.

THEOREM 4.6. *Let* $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$, *and*

$$h_\nu(f;\, z) = \sum_{k=0}^{\infty} \beta_k z^k, \tag{4.6}$$

where

$$\beta_k = (k+\nu)!\alpha_k, \tag{4.7}$$

and ν *is arbitrary. Then* $h_\nu(f;\, z)$ *has a finite nonzero radius of convergence* R *if and only if* $f(z)$ *is an entire function of order* 1 *and type* τ. *In this case* $R = 1/\tau$.

Proof. The limiting form of Stirling's formula for the factorial function is

$$\lim_{n\to\infty} \frac{e}{n}(n!)^{1/n} = 1.$$

From this it is trivial to establish that

$$\lim_{n\to\infty} \frac{e}{n}((n+\nu)!)^{1/n} = 1 \quad \text{for all finite } \nu. \tag{4.8}$$

Using this, together with (4.5) and (4.7) above, it follows that, when $h_\nu(f;\, z)$ has a finite nonzero radius of convergence R,

$$\frac{1}{R} = \limsup |\beta_n|^{1/n} = \limsup (n+\nu!)^{1/n}|\alpha_n|^{1/n} = \frac{1}{e}\limsup n|\alpha_n|^{1/n}. \tag{4.9}$$

Now, when $f(z)$ is not an entire function, it has a finite radius of convergence

$$\frac{1}{R_f} = \limsup |\alpha_n|^{1/n}, \tag{4.10}$$

and if this limit exists, the limit in (4.9) is infinite. When $f(z)$ is an entire function of order μ and type τ, we may employ (4.4) to put (4.9) in the form

$$\frac{1}{R} = \frac{1}{e}\left(\limsup n^{\mu}|\alpha_n|^{\mu/n}\right)^{1/\mu} = \frac{1}{e}(e\mu\tau)^{1/\mu}\lim n^{1-1/\mu}. \tag{4.11}$$

This leads to an infinite value for R when $\mu < 1$ and a zero value for R when $\mu > 1$. When $\mu = 1$ this gives $R = 1/\tau$, establishing the theorem. □

Naturally we now identify $f(z)$ and $h_\nu(f; z)$ in Theorem 4.6 with the functions of the same name in (1.1) and (2.1), an so identify R with $R(f; 0)$ of (2.2). This theorem establishes that $R(f; 0)$ is independent of ν.

Moreover, if we replace $f(z)$ by $f(z; p)$ and a_k by $a_k(p)$ in the statement of this theorem, it becomes necessary to replace $h_\nu(f; z)$ by $h_\nu(f; z; p)$. The result of this modified theorem includes the corollary that $R(f; p)$ of (2.2) is independent of ν.

The technique, described in Section 1, could be applied to any series. However, the theory of Section 2 requires a nonzero finite value of $R(f; 0)$. Theorem 4.6 above shows that the theory of Section 2 applies only when $f(z)$ is an entire function of order 1.

It is clearly not advantageous to use the technique when $f(z)$ is an entire function of order $\mu < 1$. In this case, unless $p = 0$, $f(z; p) = e^{-pz}f(z)$ is an entire function of order 1 and so its series converges more slowly than that of the original series $f(z)$.

When $f(z)$ is an entire function of order $\mu > 1$ and type τ or is analytic with radius of convergence R_f, $f(z; p)$ has the same characteristic, and Criterion 1.8 above yields an identical rate of convergence for $f(z)$ as for $f(z; p)$.

5. Remarks About Numerical Calculation. In practice one can envision at least two different types of application. On one hand there is a situation where one is investigating properties of special functions. For example, the sum $\Sigma z^k/kk!$ occurs in one formulation of the special function $\mathrm{Ei}(z)$. It may well be of intellectual interest to reexpress this sum. On the other hand is a situation in which the coefficients a_j are expensive or difficult to compute. In the problem which motivated this investigation a_j is a Laguerre expansion coefficient

$$\alpha_j = \int_0^\infty e^{-t} L_j(t)\phi(t)\,dt,$$

which has to be determined numerically, the calculation becoming significantly more difficult as j is increased. In a calculation of this sort, after the values α_j, $j = 0, 1, 2, \ldots, N$, have been calculated, the work involved in calculating $\alpha_j(p)$, $j = 0, 1, \ldots, N$, for several values of p may be much less than that of calculating the single further coefficient α_{N+1}. In this case we may seek to calculate a numerical approximation to p_{opt}.

We have used two approaches. One involves minimizing functions like $(\alpha_N(p))^2$ or $(\alpha_N(p))^2 + (\alpha_{N-1}(p))^2$ numerically. One may take advantage of the circumstance that $\partial\alpha_N(p)/\partial p = -\alpha_{N-1}(p)$ in such a calculation. The other involves taking advantage of the fact that p_{opt} is independent of z, taking $z = z_0$ where z_0 is small enough so that $f(z)$ may be approximated to sufficient accuracy with N terms of its expansion and minimizing either the square of

$$E_M(z_0, p) = e^{pz_0} \sum_{j=0}^{M} \alpha_j(p) z_0^j - f(z_0) \tag{5.1}$$

or a sum such as

$$\sum_{l=M}^{N} (E_l(z_0, p))^2. \tag{5.2}$$

The reader should note that the value of p which minimizes these functionals is not $p_{\mathrm{opt}}(f)$ but is a function depending on N and M and z_0. However, it may be close to p_{opt}.

In some cases these methods worked well. In others, in which the theoretical result was available, they provided only crude approximations to p_{opt}. But in some cases of practical interest we found that a value of p determined by these means was more useful than the theoretical value. That is, what we actually required was to attain as accurate an approximation as possible using only N terms. For given finite N, the value of p required is close to the minimum of (5.2), but need not be particularly close to p_{opt}.

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